28 Considérese el movimiento de un fotón en el espacio-tiempo de Schwarzschild. Razónese que éste puede moverse según una órbita circular de radio r constante. Calcúlese el radio de dicha órbita. Calcúlese el período de la misma en el tiempo coordenado. Calcúlese el período en el tiempo propio de un observador no geodésico que se encuentra en un punto de la órbita.

Being massless, the photon's velocity satisfies

$g_{\mu\nu} u^\mu u^\nu = 0$

$$-f(r)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0.$$

Since θ may be taken without loss of generality as $\theta = \frac{\pi}{2}$

and

$$f\,\frac{dt}{d\tau} = E = \text{const}$$

$$r^2\dot{\phi} = L = \text{const}$$

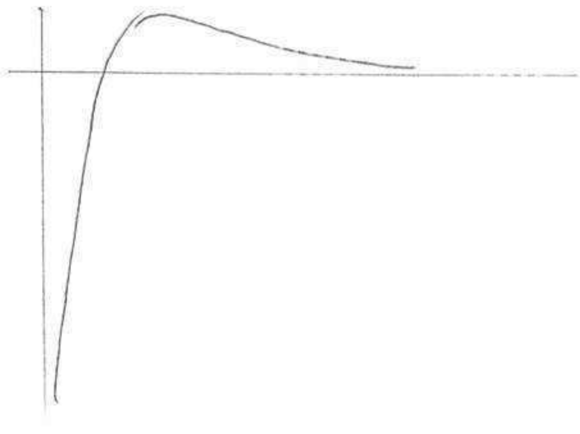
one has

$$-\frac{E^2}{f} + \frac{1}{f}\left(\frac{dr}{d\tau}\right)^2 + r^2\frac{L^2}{r^4} = 0$$

$$\left(\frac{dr}{d\tau}\right)^2 + f\,\frac{L^2}{r^2} = E^2 \iff \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2} = E^2 \longrightarrow$$

$$f = 1 - \frac{2m}{r}$$

$$V_{eff} = \frac{1}{r^2}\left(1 - \frac{2}{r}\right)$$



A circular orbit with constant r has $\frac{dr}{d\tau}=0$. Since E^2 is constant for all r, the "effective potential energy"

$$V_{eff} := \left(1-\frac{2m}{r}\right)\frac{L^2}{r^2}$$

must have a maximum. Let us find it

$$V'_{eff} = \frac{2mL^2}{r^4} - \left(1-\frac{2m}{r}\right)\frac{2L^2}{r^3} \quad \Bigg| \quad \frac{m}{r_0} = 1-\frac{2m}{r_0} \Rightarrow r_0 = 3m$$

$$V'_{eff}(r_0) = 0$$

$$V''_{eff} = -\frac{24mL^2}{r^5} + \frac{6L^2}{r^4} = \frac{6L^2}{r^4}\left(1-\frac{4m}{r}\right) \Rightarrow V''_{eff}(r_0) < 0 \Rightarrow$$

$$\Rightarrow V_{eff} \text{ has a maximum at } r_0 = 3m.$$

All in all

$$\boxed{\text{photon's circular orbit has radius } r_0 = 3m}$$

The coordinate period is

$$t_p = \int dt = \int_0^{2\pi} \frac{dt}{d\tau}\frac{d\tau}{d\phi}\,d\phi = \int_0^{2\pi} \frac{E}{1-\frac{2m}{r_0}}\,\frac{1}{\frac{L}{r_0^2}}\,d\phi = 27m^2\frac{E}{L}2\pi$$

On $r=r_0$ one has

$$r=r_0 \Rightarrow \quad E^2 = f(r_0)\frac{L^2}{r_0^2} = \left(1-\frac{2m}{r_0}\right)\frac{L^2}{r_0^2} \Rightarrow \frac{E}{L} = \frac{1}{\sqrt{3}r_0} = \frac{1}{3\sqrt{3}m}$$

Hence

$$\boxed{t_p = 6\sqrt{3}\pi m}$$

For an observer at a fixed point on the orbit, $\dot{r} = \dot{\phi} = 0$, so that (recall that for the observer $u^2 = -1$)

$$-1 = -f(r_0)\left(\frac{dt}{d\tau}\right)^2$$

$$1 = \left(1 - \frac{2m}{r_0}\right)\left(\frac{dt}{d\tau}\right)^2 \Leftrightarrow \quad d\tau = \frac{dt}{\sqrt{3}} \quad \Rightarrow \quad \tau_p = \frac{t_p}{\sqrt{3}}$$

$$\boxed{\tau_p = 6\pi m}$$