5.1

$$ds^2 = r^2(dx^2+dy^2) - 2\,du\,dr + H(r)\,du^2$$

$$g_{\mu\nu} = \begin{array}{c|c} & \begin{matrix} u & r & x & y \end{matrix} \\ \begin{matrix} u \\ r \\ x \\ y \end{matrix} & \left(\begin{array}{cc|cc} H & -1 & & \\ -1 & 0 & & \\ \hline & & r^2 & \\ & & & r^2 \end{array}\right) \end{array} \qquad g^{\mu\nu} = \left(\begin{array}{cc|cc} 0 & -1 & & \\ -1 & -H & & \\ \hline & & 1/r^2 & \\ & & & 1/r^2 \end{array}\right)$$

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda}\right)$$

$$\Gamma^u{}_{\nu\lambda} = -\frac{1}{2}\big(\partial_\nu \underbrace{g_{r\lambda}}_{\text{const}} + \partial_\lambda \underbrace{g_{\nu r}}_{\text{const.}} - \partial_r g_{\nu\lambda}\big) = \frac{1}{2}\partial_r g_{\nu\lambda}$$

$$\boxed{\Gamma^u{}_{uu} = \frac{1}{2}H'} \quad \boxed{\Gamma^u{}_{xx} = \Gamma^u{}_{yy} = r}$$

$$\Gamma^r{}_{\nu\lambda} = -\frac{1}{2}\big(\partial_\nu g_{u\lambda} + \partial_\lambda g_{\nu u} - \overset{\to 0}{\partial_u} g_{\nu\lambda}\big) \rightsquigarrow \neq 0 \text{ only for } (\nu\lambda) = (ur)$$

$$\underbrace{-\frac{1}{2}H\left(\partial_\nu g_{r\lambda} + \partial_\lambda g_{\nu r} - \partial_r g_{\nu\lambda}\right)}_{H \text{ times } \Gamma^u{}_{\nu\lambda}}$$

$$\boxed{\Gamma^r{}_{uu} = \frac{1}{2}HH'} \quad \boxed{\Gamma^r{}_{xx} = \Gamma^u{}_{yy} = rH} \quad \boxed{\Gamma^r{}_{ur} = -\frac{1}{2}H'}$$

$$\Gamma^x{}_{\nu\lambda} = \frac{1}{2r^2}\big(\partial_\nu g_{x\lambda} + \partial_\lambda g_{\nu x} - \underset{\downarrow 0}{\partial_x} g_{\nu\lambda}\big)$$

$$\boxed{\Gamma^x{}_{rx} = \Gamma^y{}_{ry} = \frac{1}{r}}$$

$$\Gamma^\alpha{}_{\alpha\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g} = \delta_\mu{}^r\,\frac{2}{r}$$

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha{}_{\mu\nu} - \partial_\nu\Gamma^\alpha{}_{\alpha\mu} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu\mu} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}$$

$$R_{u\nu} = \partial_\alpha\Gamma^\alpha{}_{u\nu} - \partial_\nu\underbrace{\Gamma^\alpha{}_{\alpha u}}_{=0} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu u} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha u}$$

$$\cancelto{0}{\partial_u\Gamma^u{}_{uu}}\,\delta_\nu{}^u + \partial_r\Gamma^r{}_{uu}\delta_\nu{}^u + \partial_r\Gamma^r{}_{ur}\delta_\nu{}^r$$

$$\cancelto{0}{\Gamma^\alpha{}_{\alpha u}}\,\Gamma^u{}_{uu}\delta_\nu{}^u + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{uu}\delta_\nu{}^u + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{ur}\delta_\nu{}^r$$

$$\Gamma^u{}_{uu}\Gamma^u{}_{uu}\delta_\nu{}^u + \cancelto{0}{\Gamma^u{}_{\nu r}}\,\Gamma^r{}_{uu} + \Gamma^r{}_{ur}\Gamma^r{}_{ru}\delta_\nu{}^u$$

$$R_{uu} = \partial_r\Gamma^r{}_{uu} + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{uu} - (\Gamma^u{}_{uu})^2 - (\Gamma^r{}_{ur})^2$$

$$\boxed{R_{uu} = \frac{1}{2}(HH')' + \frac{1}{r}HH' - \frac{1}{2}H'^2}$$

$$R_{ur} = \partial_r\Gamma^r{}_{ur} + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{ur}$$

$$\boxed{R_{ur} = -\frac{1}{2}H'' - \frac{1}{r}H'}$$

$$R_{r\nu} = \partial_\alpha\Gamma^\alpha{}_{r\nu} - \underbrace{\partial_\nu\Gamma^\alpha{}_{\alpha r}}_{=\delta_\nu{}^r\partial_r\Gamma^\alpha{}_{\alpha r} = -\frac{2}{r^2}\delta_\nu{}^r} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu r} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha r}$$

$$\partial_r\Gamma^r{}_{ru}\delta_\nu{}^u + \cancelto{0}{\partial_x}\Gamma^x{}_{rx}\delta_\nu{}^x + \cancelto{0}{\partial_y}\Gamma^y{}_{ry}\delta_\nu{}^y$$

$$\Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{ur}\delta_\nu{}^u + \cancelto{0}{\Gamma^\alpha{}_{\alpha x}}\,\Gamma^x{}_{xr}\delta_\nu{}^x + \cancelto{0}{\Gamma^\alpha{}_{\alpha y}}\,\Gamma^y{}_{yr}\delta_\nu{}^y$$

$$\cancelto{0}{\Gamma^u{}_{\nu r}}\,\Gamma^r{}_{ur} + \Gamma^x{}_{x\nu}\Gamma^x{}_{xr} + \Gamma^y{}_{y\nu}\Gamma^y{}_{yr}$$

$$R_{ru} = \partial_r \Gamma^r{}_{ru} + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{ur} \quad \text{as above}$$

$$R_{rr} = -\left(-\frac{2}{r^2}\right) - \left(\frac{1}{r^2} + \frac{1}{r^2}\right) = 0$$

$$R_{x\nu} = \underbrace{\partial_\alpha \Gamma^\alpha{}_{x\nu}} - \underbrace{\partial_\nu \Gamma^\alpha{}_{\alpha x}}_{=0} + \underbrace{\Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu x}} - \underbrace{\Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha x}}$$

$$\cancelto{0}{\partial_u \Gamma^u{}_{xx}\delta_\nu{}^x} + \partial_r \Gamma^r{}_{xx}\delta_\nu{}^x + \cancelto{0}{\partial_x}\Gamma^x{}_{xr}\delta_\nu{}^r$$

$$\cancelto{0}{\Gamma^\alpha{}_{\alpha u}}\Gamma^u{}_{xx}\delta_\nu{}^x + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{xx}\delta_\nu{}^x + \cancelto{0}{\Gamma^\alpha{}_{\alpha x}}\Gamma^x{}_{xr}\delta_\nu{}^r$$

$$\cancelto{0}{\Gamma^x{}_{\nu u}}\Gamma^u{}_{xx} + \Gamma^x{}_{xr}\Gamma^r{}_{xx}\delta_\nu{}^x + \Gamma^r{}_{xx}\Gamma^x{}_{xr}\delta_\nu{}^x$$

$$R_{xx} = \partial_r \Gamma^r{}_{xx} + \Gamma^\alpha{}_{\alpha r}\Gamma^r{}_{xx} - 2\,\Gamma^x{}_{xr}\Gamma^r{}_{xx}$$

$$R_{xx} = (rH)' + \cancel{\frac{2}{r}(rH)} - \cancel{2\frac{1}{r}rH}$$

$$\boxed{R_{xx} = R_{yy} = (rH)'}$$

Everything is symmetric under $x \leftrightarrow y$.

$$R = -2R_{ur} - HR_{rr} + \frac{1}{r^2}\left(R_{xx} + R_{yy}\right)$$

$$= H'' + \frac{2}{r}H' + \frac{2}{r^2}(rH)' = H'' + \frac{2}{r}H' + \frac{2}{r^2}H + \frac{2H'}{r}$$

$$\boxed{R = H'' + \frac{4}{r}H' + \frac{2}{r^2}H}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \qquad (4.1)$$

$(\mu\nu = uu)$
$$\frac{1}{2} \cancel{H H''} + \frac{1}{r} H H' - \frac{1}{2} H \left(\cancel{H''} + \frac{4}{r} H' + \frac{2}{r^2} H \right) + \Lambda H = 0$$

$$\frac{1}{r} H H' + \frac{H^2}{r^2} = \Lambda H$$

$$\frac{1}{r} H' + \frac{H}{r^2} = \Lambda \qquad (4.2)$$

$(\mu\nu = ur)$
$$-\frac{1}{2} \cancel{H''} - \frac{H'}{r} + \frac{1}{2} \left(\cancel{H''} + \frac{4}{r} H' + \frac{2}{r^2} H \right) - \Lambda = 0$$

$$\frac{1}{r} H' + \frac{H}{r^2} = \Lambda$$

Note that $R_{rr} = 0$ is consistent with (4.1) and $g_{rr} = 0$

$(\mu\nu = xx)$
$(\mu\nu = yy)$
$$(rH)' - \frac{1}{2} r^2 \left(H'' + \frac{4}{r} H' + \frac{2}{r^2} H \right) + r^2 \Lambda = 0$$

$$\cancel{H} + rH' - \frac{r^2}{2} H'' - 2rH' - \cancel{H} + r^2 \Lambda = 0$$

$$\frac{1}{2} H'' + \frac{H'}{r} = \Lambda \qquad (4.3)$$

Solution of (4.2):

Homogeneous: $H' = -\frac{H}{r} \Rightarrow H = \frac{h_0}{r}$

Complete $H = \frac{h(r)}{r} \Rightarrow H' = \frac{h'}{r} - \frac{h}{r^2}$

$$\frac{h'}{r^2} - \cancel{\frac{h}{r^3}} + \cancel{\frac{h}{r^3}} = \Lambda$$

$$h = \frac{\Lambda}{3} r^3 + \text{const}$$

$$H(r) = \frac{2m_0}{r} + \frac{\Lambda}{3} r^2 \tag{5.1}$$

Check: (5.1) solves (4.3)

$$H' = -\frac{2m_0}{r^2} + \frac{2\Lambda}{3} r$$

$$H'' = \frac{4m_0}{r^3} + \frac{2\Lambda}{3}$$

$$\frac{1}{2} H'' + \frac{H'}{r} = \cancel{\frac{2m_0}{r^3}} + \frac{\Lambda}{3} - \cancel{\frac{2m_0}{r^3}} + \frac{2\Lambda}{3} = \Lambda \quad \checkmark$$