UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FISICAS

6 de Septiembre de 2017

Examen Final de Física Computacional

Nombre y Apellidos:

DNI y Firma:

Advertencias: La calculadora en modo de **radianes**. Sus cuentas deben justificar las respuestas que usted escriba.

1. [2.5 puntos] Sea
$$F(x,y) = \begin{pmatrix} x^2 + y^2 - x \\ x^2 - y^2 - y \end{pmatrix}.$$

   a) Hacer un dibujo de $F(x,y) = 0$ para ver dónde se encuentran las soluciones. (Es obvio que $x = y = 0$ es una solución. Esta ya se la digo yo. Pero hay más)

   b) Escribir explícitamente el método de Newton-Raphson para resolver $F(x,y) = 0$ calculando la matriz Jacobiana asociada, así como su inversa (cuando exista).

   c) Tomando como punto incial $(x_0, y_0) = (1/2, 1/2)$, encontrar las tres primeras aproximaciones $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ proporcionadas por dicho método. Trabaje en aritmética exacta, por ejemplo, escriba 5/22 y no 0.22727.

   d) ¿Se aproxima esta secuencia de tres puntos a la solución $(0,0)$ o a otra solución?

   e) Convenza al examinador mediante un argumento gráfico o algebraico o como usted considere oportuno de que el número de raíces **complejas** del problema es: 1) ninguna, 2) una, 3) dos, 4) tres... Conteste lo que corresponda.

2. [2.25 puntos] Sea la función $f(x) = (1 + x^2)\, e^x$.

   a) Usar la fórmula de diferenciación numérica centrada
$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h},$$
   con steps $h = 0.2, h = 0.05, h = 0.01$ para aproximar $f'(0)$. A los valores obtenidos los llamaremos d_1, d_2, d_3, respectivamente.

   b) Con los datos d_1, d_2, d_3 calcular la mejor aproximación posible (extrapolación de Richardson) a $f'(0)$.

   c) El error de esta aproximación será de la forma $\mathcal{O}(h^p)$ para un cierto p. ¿Cuál es el valor de p?

   d) Aproximadamente, qué step h debería utilizarse en la fórmula centrada para obtener un valor igual al de Richardson? (este apartdo es el que indica lo buenísima que es esta extrapolación como algoritmo matemático).

3. [2.25 puntos] a) Usando técnicas simples de Monte Carlo evaluar la integral
$$\int\int_{\Omega} \sin \sqrt{\log(x+y+1)}\, dxdy,$$

donde log es logaritmo neperiano (base e) y Ω es el círculo definido por la desigualdad

$$\Omega = \left\{(x, y) : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \frac{1}{4}\right\}$$

Para ello se han tirado aleatoriamente los $N = 20$ puntos (x, y) que siguen

$$\begin{array}{l}
(0.338, 0.152),\ (0.0654, 0.367),\ (0.655, 0.115),\ (0.570, 0.827),\\
(0.374, 0.114),\ (0.387, 0.380),\ (0.413, 0.0546),\ (0.390, 0.986),\\
(0.231, 0.424),\ (0.954, 0.413),\ (0.873, 0.260),\ (0.178, 0.347),\\
(0.129, 0.196),\ (0.812, 0.853),\ (0.625, 0.767),\ (0.584, 0.131),\\
(0.742, 0.478),\ (0.872, 0.782),\ (0.868, 0.921),\ (0.0724, 0.555).
\end{array}$$

Estos puntos están distribuidos uniformemente en el cuadrado $(0, 1) \times (0, 1)$.

b) Calcule la desviación standard como viene en el libro de Press (recuerde que lo vimos en clase) e interprete el resultado con los intevalos de confianza a 1, 2 y 3 desviaciones.

c) A la vista del resultado obtenido en b), podría el valor exacto de la integral, que geométricamente es un volumen, ser 0.3?

4. [0.5 puntos] a) ¿Qué calcula el método iterativo $x_{n+1} = 2x_n - x_n\, y$, sabiendo que y es una constante no nula? b) ¿Podría tratarse de un *método de Newton*? c) Si lo es, identifique la función $f(x)$ de la que se calculan los ceros.

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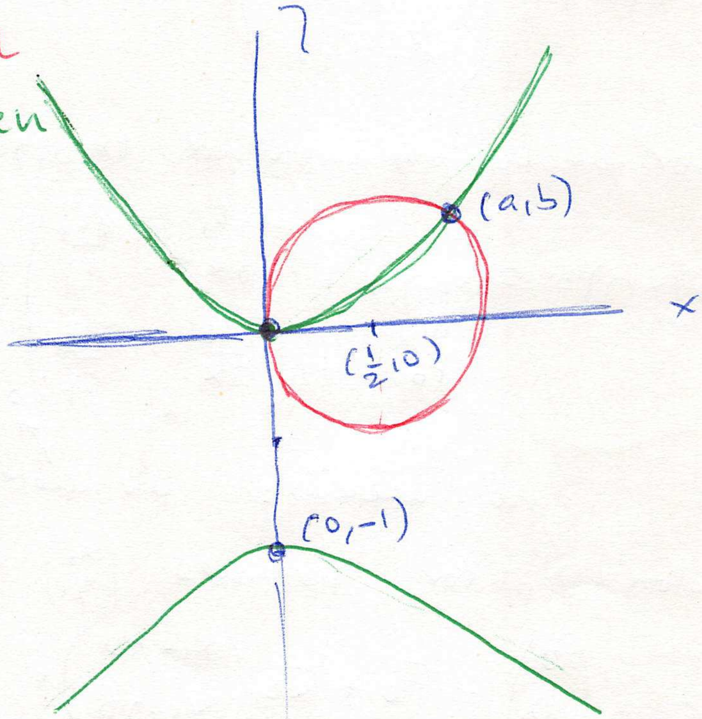
① $\begin{cases} x^2+y^2=x & \text{red} \\ x^2-y^2=y & \text{green} \end{cases}$



a) The curve $x^2+y^2=x$ is the circumference $\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$, as one easily checks. It is centered at $\left(\frac{1}{2},0\right)$ and has radius $\frac{1}{2}$. Similarly, $x^2-y^2=y$ is the hyperbola $x^2-\left(y+\frac{1}{2}\right)^2=-\frac{1}{4}$ (same trick as before), or $\left(y+\frac{1}{2}\right)^2-x^2=\frac{1}{4}$, centered at $(0,-1/2)$.

$x=0$ $\quad \left(y+\frac{1}{2}\right)^2=\frac{1}{4}, \quad y=-\frac{1}{2}\pm\frac{1}{2} \begin{cases} 0 & : (0,0) \\ -1 & : (0,-1) \end{cases}$

This is to draw the hyperbola...

trivial

The problem has two real roots, one is (0,0) and the other is (a,b) [see the picture above]

No need to say that we graphically deduce that

$$\frac{1}{2}<a<1, \quad 0<b<\frac{1}{2}.$$

With the Newton-Raphson algorithm calculate (a,b).

b) c) $f_1(x,y) \equiv x^2+y^2-x \quad , \quad f_2(x,y) \equiv x^2-y^2-y$

$$F(x,y) = \begin{pmatrix} x^2+y^2-x \\ x^2-y^2-y \end{pmatrix}$$

Taylor expansion around (x_0,y_0) is

$$F(x,y) = F(x_0,y_0) + \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_0 \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \cdots$$

quadratic terms in $(x-x_0)$, $(y-y_0)$.

evaluated at (x_0,y_0)

this means:

Newton-Raphson defines $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ as the point that satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = F_0 + J_0 \begin{pmatrix} x_1-x_0 \\ y_1-y_0 \end{pmatrix}.$$

Here,

$F_0 \equiv F(x_0,y_0)$

$J \equiv \text{Jacobian} \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$

$J_0 \equiv$ Jacobian evaluated at (x_0,y_0)

and, by construction,

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J_0^{-1} F_0 ,$$

This is the main equation of the algorithm.

In our particular example,

$$J = \begin{pmatrix} 2x-1 & 2y \\ 2x & -2y-1 \end{pmatrix}$$

$$J^{-1} = \begin{pmatrix} 2y+1 & 2y \\ 2x & -2x+1 \end{pmatrix} \frac{1}{[8xy+2x-2y-1]}$$

$\det J = -8xy+2x+2y+1$:

[A result that you must remember (without calculations!!! 2x2 is easy!!)]

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: the inverse of this matrix is

"almost" $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ [diagonal is flipped, non diagonal is negated]

I say "almost" because the determinant is also present:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \underbrace{(ad-bc)}_{\text{determinant of } \begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Conclusion: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$

]

Starting with $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ we calculate $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$.

The calculations follow:

$$J_0 = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}, \quad J_0^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} - \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1/2 \end{pmatrix}}_{= \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \quad : \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - J_0^{-1}F_0 .$$

If $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$, the next point $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 13/16 \\ 7/16 \end{pmatrix} = \begin{pmatrix} 0.8125 \\ 0.4345 \end{pmatrix}$

because

$$J_1 = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \text{ and } J_1^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$: flipped and negated.

and

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - \underbrace{\frac{1}{4}\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}}_{= \begin{pmatrix} 3/16 \\ 1/16 \end{pmatrix}} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 3/16 \\ 1/16 \end{pmatrix} = \begin{pmatrix} \frac{13}{16} \\ \frac{7}{16} \end{pmatrix}$$

With $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 13/16 \\ 7/16 \end{pmatrix}$ calculate $\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{2055}{2656} \\ \frac{1117}{2656} \end{pmatrix} \approx \begin{pmatrix} 0.\underline{77}371988 \\ 0.\underline{42}055723 \end{pmatrix}$

exact digits

as follows

$$J_2 = \begin{pmatrix} 10/16 & 14/16 \\ 26/16 & -30/16 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 10 & 14 \\ 26 & -30 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 5 & 7 \\ 13 & -15 \end{pmatrix}$$

$$J_2^{-1} = \begin{pmatrix} -15/8 & -7/8 \\ -13/8 & 5/8 \end{pmatrix}\left[-\frac{8^2}{166}\right], \quad \det J_2 = -\frac{166}{8^2}$$

$$= \frac{8}{166}\begin{pmatrix} 15 & 7 \\ 13 & -5 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} \frac{13^2}{16^2} + \frac{7^2}{16^2} - \frac{13}{16} \\ \frac{13^2}{16^2} - \frac{7^2}{16^2} - \frac{7}{16} \end{pmatrix} = \frac{1}{16^2}\begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 13/16 \\ 7/16 \end{pmatrix} - \frac{8}{166} \cdot \frac{1}{16 \cdot 16}\begin{pmatrix} 15 & 7 \\ 13 & -5 \end{pmatrix}\begin{pmatrix} 10 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 13/16 \\ 7/16 \end{pmatrix} - \frac{\cancel{8}}{166} \cdot \frac{1}{\cancel{8} \cdot 16}\begin{pmatrix} 15 & 7 \\ 13 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 13 - (15 \cdot 5 + 7 \cdot 4)/166 \\ 7 - (13 \cdot 5 - 20)/166 \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 13 - \frac{103}{166} \\ 7 - \frac{45}{166} \end{pmatrix} = \begin{pmatrix} \frac{2055}{2656} \\ \frac{1117}{2656} \end{pmatrix}.$$

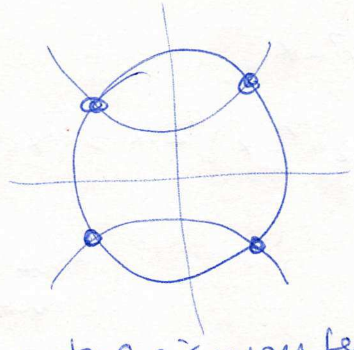
d)

Maple says that the exact solution is $(a, b) =$
$= (0.\underline{77}18445063,\ 0.\underline{41}96433776)$

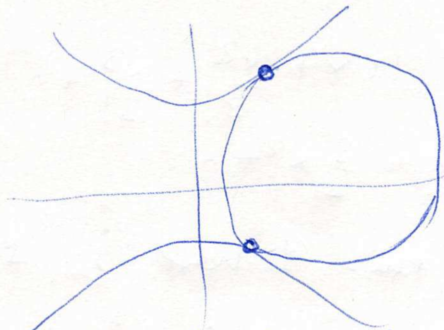
The sequence $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ approaches (a, b), not $(0,0)$ as a few iterations show.

d) How many complex solution's does the problem have? Real: two

Graphically: Think of this, similar but not equal, problem: it has four real solutions

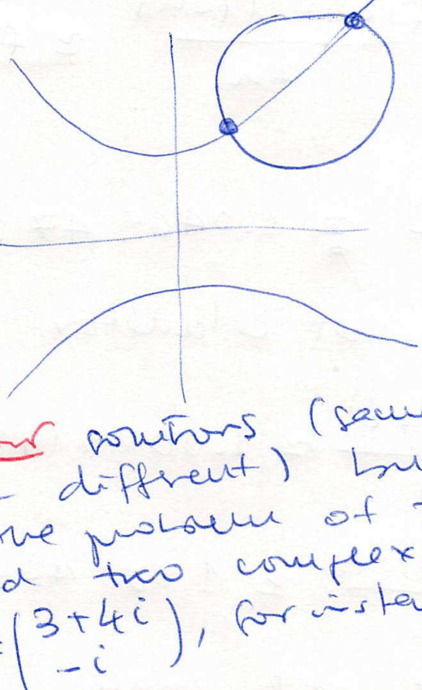


Translate now the circumference... it has two real



solutions (two points) but each solution is of multiplicity two, that is, it has four real solutions.

Translate again the circumference



it has also four solutions (same as in the other cases... no reason to be different) but two are complex solutions. In the problem of the exam: two real solutions and two complex. To find the complex try with $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3+4i \\ -i \end{pmatrix}$, for instance.

These are: $(0.11407\ldots + 0.55757\ldots\, i,\ -0.70882\ldots - 0.303\ldots\, i)$

and the complex conjugate

... los puntos suspensivos es que no preciso más...

Algebraically: the system $\begin{cases} x^2+y^2=x \\ x^2-y^2=y \end{cases}$

reduces to a polynomial of order 4 in x (or in y):

$$\begin{array}{l} x^2+y^2=x \\ x^2-y^2=y \\ \hline 2x^2 = x+y \;\Rightarrow\; y = 2x^2-x \end{array}$$

$$0 = x^2+y^2-x = x^2+(2x^2-x)^2-x = x\,(4x^3-4x^2+2x-1)$$

two real: $x=0$, $x^3+\dots$
two complex!

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② $f(x) = (1+x^2)\,e^x = (1+x^2)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)$

$= 1 + x + \frac{3}{2}x^2 + \dots$

this 1 here is $f'(0)$

Then: we want to calculate $f'(0)$ whose exact value is 1. We use $h_1 = 0.2$, $h_2 = 0.05$, $h_3 = 0.01$ in the difference formula

$$\frac{f(h)-f(-h)}{2h}$$

and obtain

$$d_1 \equiv \frac{f(h_1)-f(-h_1)}{2h_1} \simeq 1.046947212$$

good pocket calculator

$$d_2 \equiv \frac{f(h_2)-f(-h_2)}{2h_2} \simeq 1.002917759$$

$$d_3 \equiv \frac{f(h_3)-f(-h_3)}{2h_3} \simeq 1.000116665$$

obviously, d_3 is the best value (closest to 1), then d_2 and d_1!

b) Richardson extrapolation:

$$\frac{f(h)-f(-h)}{2h}$$ is an **even** function of h

(change h by $-h$ and you will see it). Then

will be $1 = f'(0)$

Key formula: $$\frac{f(h)-f(-h)}{2h} = a_0 + a_2h^2 + a_4h^4 + a_6h^6 + \dots$$

To note what follows the values of $a_2, a_4, a_6 \dots$ do not matter. Take $h_1 = h$, $h_2 = \frac{h}{4}$, $h_3 = \frac{h}{20}$

and

$$d_1 = a_0 + a_2h^2 + a_4h^4 + a_6h^6 + \dots$$

$$d_2 = a_0 + a_2\left(\frac{h}{4}\right)^2 + a_4\left(\frac{h}{4}\right)^4 + a_6\left(\frac{h}{4}\right)^6 + \dots$$

$$d_3 = a_0 + a_2\left(\frac{h}{20}\right)^2 + a_4\left(\frac{h}{20}\right)^4 + a_6\left(\frac{h}{20}\right)^6 + \dots$$

and obtain

lo he calculado pero no lo uso.

$$d_1, d_2 \to \frac{16d_2-d_1}{15} = a_0 - \frac{a_4h^4}{16} - \frac{17}{256}a_6h^6 - \dots$$

$$d_2, d_3 \to \frac{25d_3-d_2}{24} = a_0 - \frac{a_4}{6400}h^4 + (\dots)h^6 + \dots$$

¡no lo calculo. ¡y así: no lo uso.

$$\frac{6250d_3 - 266d_2 + d_1}{5985} = a_0 + \frac{a_6}{6400}h^6 + \dots$$

does not matter a_6 value. (except if zero... but here it is not the case)

Solution of the problem: The best value obtained with d_1, d_2, d_3 is not d_3 but

$$\frac{6250 d_3 - 266 d_2 + d_1}{5985} ; \quad O(h^6)$$

↑ move

$$\frac{16 d_2 - d_1}{15} = a_0 - \frac{a_4}{16} h^4 + \dots$$

$$\frac{25 d_3 - d_2}{24} = a_0 - \frac{a_4}{6400} h^4 + \dots$$

$$6400 \left[\frac{25 d_3 - d_2}{24} \right] - 16 \left[\frac{16 d_2 - d_1}{15} \right] = 6384 a_0 + \frac{399}{400} a_6 h^6 + \dots$$

Now we compute

$$\frac{6250 d_3 - 266 d_2 + d_1}{5985} \simeq 0.\underbrace{99999999}_{8 \text{ nueves}}69$$

¿Saben ustedes que $0.9999999\overline{9} \dots = 1$, no?

Esto es casi 1.

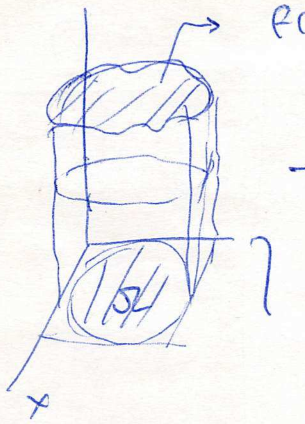
$$|1 - 0.9999999969| = 3.1 \times 10^{-9}$$

d) To obtain this precision (3.1×10^{-9}) we have to take $h \in [0.00005, 0.00006]$ for the formula

$$\frac{f(h) - f(-h)}{2h}.$$

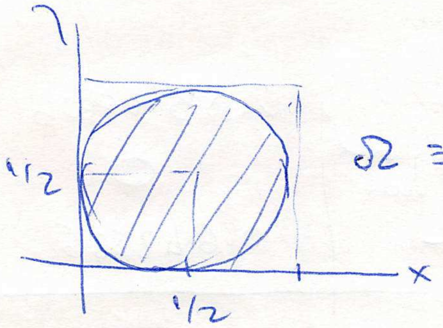
Sensacional!!!

(3)



$$f(x,y) = \sin\sqrt{\log(x+y+1)}$$

→ it is the volume of this "cylinder" what $I \equiv \int_\Omega dx \int dy\, f(x,y)$ calculates.



$\Omega \equiv$ circle (disk) centred at $(\frac{1}{2},\frac{1}{2})$ and radius $\frac{1}{2}$

In our case, I, the integral to calculate is bounded by the volume of a cylinder of height 1, because $|\sin(\cdot)| \le 1$. In other words

$$|I| \le \underbrace{\frac{\pi}{4}}_{\text{area of }\Omega} \times 1 \simeq 0.7854;$$

[I do not know the exact value of I, only that $|I| \le \frac{\pi}{4}$.]

x	y	in Ω?	$\sin\sqrt{\log(x+y+1)}$	
0.338	0.152	yes	0.590346	$\equiv f_1$
0.0654	0.367	yes	0.564196	$\equiv f_2$
0.655	0.115	yes	0.685749	$\equiv f_3$
0.570	0.827	yes	⋮	⋮
0.374	0.114	yes	⋮	⋮
0.387	0.380	yes	⋮	
0.413	0.0546	✓	⋮	
0.390	0.986	✓	⋮	
0.231	0.424	✓	⋮	
0.954	0.413	✓	⋮	

x	y	in Ω?	f(x,y)
0.873	0.260		⋮
0.178	0.347		⋮
0.129	0.196		⋮
0.812	0.853		⋮
0.625	0.767		⋮
0.584	0.131		⋮
0.742	0.478		⋮
0.872	0.782		0.834906
0.868	0.921	no - the only one that is not in Ω	0
0.0724	0.555		0.642566 $\equiv f_{20}$

$$\sum_1^{20} f_i = 13.195860$$

with $f_{19}=0$

$$\sum_1^{20} f_i^2 = 9.364148$$

there are several methods. Here all seems diffenet because afford different results (what do you expect if N=20!!!!!!!. Take $N=10^{10}$ and they will afford the same, more or less, number!)

<u>One method</u>

$$I_{MC} \sim \underset{\frac{\pi}{4}}{\underset{\shortparallel}{\Delta_{box}}} \times \frac{f_1+f_2+\dots+f_{18}+0+f_{20}}{19} = 0.545474$$

$$\sigma = \underset{\frac{\pi}{4}}{\underset{\shortparallel}{\Delta_{box}}} \times \sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{19}} = 0.018457$$

$$\langle f^2\rangle = \frac{1}{19}\left[f_1^2+\dots+0^2+f_{20}^2\right]$$

$$\langle f\rangle = \frac{1}{19}\left[f_1+\dots+f_{18}+0+f_{20}\right]$$

$f_{19}=0$

Compute $[I_{mc}-\sigma, I_{mc}+\sigma] = [0.52707, 0.563931]$

$[I_{mc}-2\sigma, I_{mc}+2\sigma] = [0.508559, 0.582389]$

$[I_{mc}-3\sigma, I_{mc}+3\sigma] = [0.490102, 0.600846]$

looking at this interval: The exact value of I is in $[0.5, 0.6]$... we do not know more... [with $N=20$!!!!!]

<u>Another method</u>: Exactly as the previous one, but changes

$A_{box} = \quad 1$

area of $\square$ 1

19: pts inside Ω

20: all points.

(this is when you don't know the area Ω... here was not the case... but it is almost always the case!)

$$I_{mc} \approx \underset{\overset{\parallel}{1}}{A_{box}} \times \frac{f_1 + f_2 + \dots + \overset{\overset{0}{\parallel}}{f_{19}} + f_{20}}{20} = 0.659793$$

$$\sigma = \underset{\overset{\parallel}{1}}{A_{box}} \sqrt{\frac{\langle f^2\rangle - \langle f\rangle^2}{20}} = 0.04055$$

$$\langle f^2\rangle = \frac{1}{20}\left[f_1^2 + \dots + \underset{\overset{\parallel}{0}}{f_{19}^2} + f_{20}^2\right] = \dots$$

$$\langle f\rangle = \frac{1}{20}\left[f_1 + \dots + \underset{\overset{\parallel}{0}}{f_{19}} + f_{20}\right]$$

$[I_{mc}-\sigma, I_{mc}+\sigma] = [0.619246, 0.7003396]$

$[I_{mc}-2\sigma, I_{mc}+2\sigma] = [0.578699, 0.740886]$

$[I_{mc}-3\sigma, I_{mc}+3\sigma] = [0.538153, 0.781433]$

parece distinto a este ??

No:... mirad este método con $N = 100000$

$[I_{mc}-\sigma$

$[I_{mc}-2\sigma, I_{mc}+2\sigma] =$

$[I_{mc}-3\sigma, I_{mc}+3\sigma] = [0.564245, 0.570048]$ ... ya es otra cosa...

Seguramente $I_{exact} \in [0.56, 0.57]$.

④ El esquema iterativo $x_{n+1} = 2x_n - x_n^2 y$ calcula $p=0$ y $p = \frac{1}{y}$. Obviamente $p=0$ no tiene sentido, $\sqrt{2}$ lo tiene, π, e^{-1}, $\log 2$, pero 0 no. What has sense is $1/y$, the inverse of y.

a) The scheme calculates, in principle, the fixed point p given by

$$p = 2p - p^2 y$$

$$0 = p - p^2 y = p(1 - py)$$

$p=0$ no interest

$\boxed{p = \frac{1}{y}}$

b) It might become from a Newton method if $F'(p) = 0$ in $x_{n+1} = F(x_n)$. We check this necessary condition:

$$F(x) \equiv 2x - x^2 y$$

$$F'(x) = 2 - 2xy \quad , \quad F'(p) = 2 - 2py = 0 \qquad p \equiv \frac{1}{y}$$

Yes. It might come from a Newton method. It will do if

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for a given $f(x)$.

c) ... a checking first ...

Take $f(x) = x - \frac{1}{y}$, $f'(x) = 1$, $x - \frac{f(x)}{f'(x)} = x - (x - \frac{1}{y}) = \frac{1}{y}$.

Obviously, this is not the case for

$$x - \frac{f(x)}{f'(x)} \overset{!}{=} 2x - x^2 y$$

and not $\frac{1}{y}$. Take now $f(x) = \frac{1}{x} - y$, $f'(x) = -\frac{1}{x^2}$

and $x - f/f' = x + \frac{1/x - y}{1/x^2} = x + x - x^2 y = 2x - x^2 y$.

BINGO.

$$\boxed{f(x) = \frac{1}{x} - y}$$

[In case our guessings fail one can always address the problem as follows (more difficult though!)

$$x - \frac{f(x)}{f'(x)} = 2x - x^2 y$$

or

$$-\frac{f(x)}{f'(x)} = x - x^2 y$$

numerator ≡ derivative of denominator

or

$$\frac{f'}{f} = \frac{1}{x^2 y - x} = \frac{2xy-1}{x^2 y - x} + \frac{-2xy+2}{x^2 y - x}$$

$$\frac{-2xy+2}{x^2 y - x} = \frac{2(1-xy)}{x(xy-1)}$$

starting ...

$$= \frac{2xy-1}{x^2 y - x} - \frac{2}{x}.$$

then,

$$\int \frac{df}{f} = \int dx \left[\frac{2xy-1}{x^2 y - x} - \frac{2}{x} \right] \Rightarrow$$

$$\log f = \log(x^2 y - x) - 2 \log x$$

$$= \log \left(\frac{x^2 y - x}{x^2} \right)$$

$$f = \frac{x^2 y - x}{x^2} = y - \frac{1}{x}.$$

$$\boxed{f(x) = y - \frac{1}{x}.}$$

Bingo again !!!

$$x_{n+1} = F(x_n) \quad , \quad p = 1/y$$

Para mi: $F(x) = F(p) + F'(p)(x-p) + \frac{F''(p)}{2!}(x-p)^2 + \ldots$

$$x_{n+1} = p + 0 \cdot (x-p) - \frac{2y}{2!} e_n^2 + 0 + 0 + 0 \ldots$$

$F(p) = p$

$F(x) = 2x - x^2 y$, $F'(x) = 2 - 2xy$, $F''(x) = -2y$

$F'''(x) = F^{(4)} = \ldots = 0$

$$\boxed{e_{n+1} = -y e_n^2}.$$